Q. P. Code:

(2½ Hours)

[Total Marks: 75]

N. B.: (1) **All** questions are **compulsory**.
(2) Make **suitable assumptions** wherever necessary and **state the assumptions** made.
(3) Answers to the **same question** must be **written together**.
(4) Numbers to the **right** indicate **marks**.
(5) Draw **neat labeled diagrams** wherever **necessary**.
(6) Use of **Non-programmable** calculators is **allowed**.

1. Attempt *any three* of the following: **15**

a. Define Universal Existential Statement and Existential Universal Statement. Give examples of each.

Solution: A ***universal existential statement*** is a statement that is universal because its first part says that a certain property is true for all objects of a given type, and it is existential because its second part asserts the existence of something.

e.g.

All real numbers have additive inverses.

Or: For all real numbers r , there is an additive inverse for r .

Or: For all real numbers r, there is a real number s such that s is an additive inverse for r.

An ***existential universal statement*** is a statement that is existential because its first part asserts that a certain object exists and is universal because its second part says that the object satisfies a certain property for all things of a certain kind.

Some positive integer is less than or equal to every positive integer.

Or: There is a positive integer m that is less than or equal to every positive integer.

Or: There is a positive integer m such that every positive integer is greater than or equal to m.

Or: There is a positive integer m with the property that for all positive integers $n, m \leq n$.

b. Define Cartesian product. Let **R** denote the set of all real numbers. Describe **R x R**.

Solution: Given sets A and B, the **Cartesian product of *A* and *B***, denoted $\boldsymbol{A} \times \boldsymbol{B}$ and read "A cross B," is the set of all ordered pairs (a, b), where a is in A and b is in B.

Symbolically:

$\boldsymbol{A} \times \boldsymbol{B} = \{(a, b) \mid a \in A \text{ and } b \in B\}.$

$\mathbf{R} \times \mathbf{R}$ is the set of all ordered pairs (x, y) where both x and y are real numbers. If horizontal and vertical axes are drawn on a plane and a unit length is marked off, then each ordered pair in $\mathbf{R} \times \mathbf{R}$ corresponds to a unique point in the plane, with the first and second elements of the pair indicating, respectively, the horizontal and vertical positions of the point. The term **Cartesian plane** is often used to refer to a plane with this coordinate system, as illustrated in Figure:

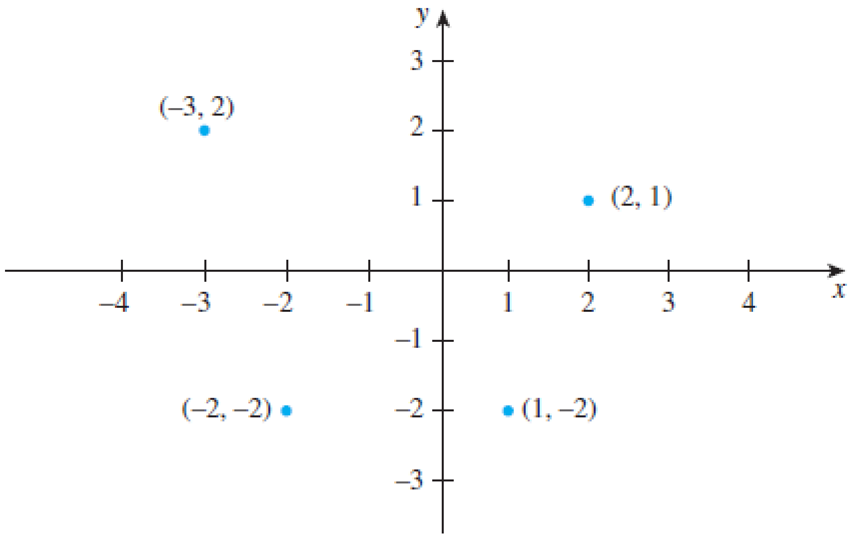


Figure: A Cartesian Plane

c. Find the number of integers between 1 and 250 that are divisible by 2 or 3 or 5 or 7.

Ans: **193**

d. Prove that $(A \cup B) \cap (A \cap B)' = (A - B) \cup (B - A)$

Solution $LHS = (A \cup B) \cap (A \cap B)'$

$= (A \cup B) \cap (A' \cup B')$ De Morgan's Law

$= [A \cap (A' \cup B')] \cup [B \cap (A' \cup B')]$ Distributive Law

$= [(A \cap A') \cup (A \cap B')] \cup [(B \cap A') \cup (B \cap B')]$ Distributive Law

$= [\emptyset \cup (A \cap B')] \cup [\emptyset \cup (B \cap A')]$

$= (A \cap B') \cup (B \cap A')$

$= (A - B) \cup (B - A)$

$= RHS$

e. Write the negation of each of the following statements as simply as possible:

i. If she works, she will earn money.
ii. He swims if and only if the water is warm.
iii. If it snows, then they do not drive the car.
iv. John is 6 feet tall and he weighs at least 120 kg.
v. The train was late or Amol's watch was slow.

Solution
i. She works or she will not earn money.
ii. He swims if and only if the water is not warm.
iii. It snows and they drive the car.
iv. John is not 6 feet tall or he weighs less than 120 kg.
v. The train was not late and Amol's watch was not slow.

f. Define the following:

i. Argument, Premises
ii. Syllogism
iii. Explain Modus Ponens and Modus Tollens with examples.

Solution
i. An **argument** is a sequence of statements, and an **argument form** is a sequence of statement forms. All statements in an argument and all statement forms in an argument form, except for the final one, are called **premises** (or **assumptions** or **hypotheses**). The final statement or statement form is called the **conclusion.** The symbol ∴, which is read "therefore," is normally placed just before the conclusion.

ii. An argument form consisting of two premises and a conclusion is called a **syllogism.** The first and second premises are called the **major premise** and **minor premise**, respectively.

iii. The most famous form of syllogism in logic is called **modus ponens.** It has the following form:

$$\text{If } p \text{ then } q.$$
$$p$$
$$\therefore\ q$$

Here is an argument of this form:
If the sum of the digits of 371,487 is divisible by 3,
then 371,487 is divisible by 3.
The sum of the digits of 371,487 is divisible by 3.
∴ 371,487 is divisible by 3.

The term modus ponens is Latin meaning "method of affirming" (the conclusion is an affirmation).

Another valid argument form is called **modus tollens.** It has the following form:

$$\text{If } p \text{ then } q.$$
$$\sim q$$
$$\therefore\ \sim p$$

Here is an example of modus tollens:
If Zeus is human, then Zeus is mortal.
Zeus is not mortal.
∴ Zeus is not human.

Modus tollens is Latin meaning "method of denying" (the conclusion is a denial).

2. Attempt *<u>any three</u>* of the following: **15**

a. Let

Q(n) be "*n* is a factor of 8,"
R(n) be "*n* is a factor of 4,"
S(n) be "$n < 5$ and $n \neq 3$,"

and suppose the domain of *n* is $\mathbf{Z}^+$, the set of positive integers. Use the ⇒ and ⇔ symbols to indicate true relationships among *Q(n), R(n),* and *S(n).*

Solution

1. The truth set of $Q(n)$ is $\{1, 2, 4, 8\}$ when the domain of *n* is $\mathbf{Z}^+$.

The truth set of $R(n)$ is $\{1, 2, 4\}$. Thus it is true that every element in the truth set of $R(n)$ is in the truth set of $Q(n)$, or, equivalently, $\forall n$ in $\mathbf{Z}^+, R(n) \to Q(n)$. So $R(n) \Rightarrow Q(n)$, or, equivalently

n is a factor of 4 ⇒ *n* is a factor of 8.

2. The truth set of $S(n)$ is $\{1, 2, 4\}$, which is identical to the truth set of $R(n)$, or, equivalently,

$\forall n$ in $\mathbf{Z}^+, R(n) \leftrightarrow S(n)$. So $R(n) \Leftrightarrow S(n)$, or, equivalently,

n is a factor of 4 ⇔ $n < 5$ and $n \neq 3$.

Moreover, since every element in the truth set of $S(n)$ is in the truth set of $Q(n)$, or, equivalently, $\forall n$ in $\mathbf{Z}^+$, $S(n) \to Q(n)$, then $S(n) \Rightarrow Q(n)$, or, equivalently,

$n < 5$ and $n \neq 3 \Rightarrow n$ is a factor of 8.

b. Define *necessary and sufficient conditions* and *only if* as applied to universal conditional statements. Rewrite the following statements as formal and informal quantified conditional statements. Do not use the word necessary or sufficient.

i. Squareness is a sufficient condition for rectangularity.

ii. Being at least 35 years old is a necessary condition for being President of the United States.

Solution

• "$\forall x$, $r(x)$ is a **sufficient condition** for $s(x)$" means "$\forall x$, if $r(x)$ then $s(x)$."

• "$\forall x$, $r(x)$ is a **necessary condition** for $s(x)$" means "$\forall x$, if $\sim r(x)$ then $\sim s(x)$" or, equivalently, "$\forall x$, if $s(x)$ then $r(x)$."

• "$\forall x$, $r(x)$ **only if** $s(x)$" means "$\forall x$, if $\sim s(x)$ then $\sim r(x)$" or, equivalently, "$\forall x$, if $r(x)$ then $s(x)$."

i. A formal version of the statement is

$\forall x$, if x is a square, then x is a rectangle.

Or, in informal language:

If a figure is a square, then it is a rectangle.

ii. Using formal language, you could write the answer as

$\forall$ people x, if x is younger than 35, then x cannot be President of the United States.

Or, by the equivalence between a statement and its contrapositive:

$\forall$ people x, if x is President of the United States, then x is at least 35 years old.

c. A college cafeteria line has four stations: salads, main courses, desserts, and beverages. The salad station offers a choice of green salad or fruit salad; the main course station offers spaghetti or fish; the dessert station offers pie or cake; and the beverage station offers milk, soda, or coffee. Three students, Uta, Tim, and Yuen, go through the line and make the following choices:

Uta: green salad, spaghetti, pie, milk

Tim: fruit salad, fish, pie, cake, milk, coffee

Yuen: spaghetti, fish, pie, soda

Write each of following statements informally and find its truth value.

i. $\exists$ an item I such that $\forall$ students S, S chose I.

ii. $\exists$ a student S such that $\forall$ items I, S chose I.

iii. $\exists$ a student S such that $\forall$ stations Z, $\exists$ an item I in Z such that S chose I.

iv. $\forall$ students S and $\forall$ stations Z, $\exists$ an item I in Z such that S chose I.

Solution

i. There is an item that was chosen by every student. This is true; every student chose pie.

ii. There is a student who chose every available item. This is false; no student chose all nine items.

iii. There is a student who chose at least one item from every station. This is true; both Uta and Tim chose at least one item from every station.

iv. Every student chose at least one item from every station. This is false; Yuen did not choose a salad.

d. Define a prime number and composite number. Give symbolic definitions of the same. Disprove the following by giving two counter examples:

i. For all real numbers a and b, if $a < b$ then $a^2 < b^2$.
ii. For all integers n, if n is odd then $(n – 1)/2$ is odd.
iii. For all integers m and n, if $2m + n$ is odd then m and n are both odd.

Solution An integer n is **prime** if, and only if, $n>1$ and for all positive integers r and s, if $n=rs$, then either r or s equals n. An integer n is **composite** if, and only if, $n>1$ and $n=rs$ for some integers r and s with $1<r<n$ and $1<s<n$.

In symbols:

n is prime $\Leftrightarrow$ $\forall$positive integers r and s, if $n = rs$ then either $r = 1$ and $s = n$ or $r = n$ and $s = 1$.

n is composite $\Leftrightarrow$ $\exists$positive integers r and s such that $n = rs$ and $1 < r < n$ and $1 < s < n$.

i. a = – 2 , b = 1; a = – 3 , b = 2 (Any values can be taken)
ii. n=5; n=7 (Any values can be taken)
iii. m=2, n=1; m=4, n=3 (Any values can be taken)

e.

Solution If n and d are integers and d _= 0 then
n is **divisible by** d if, and only if, n equals d times some integer.
Instead of "n is divisible by d," we can say that
n **is a multiple of** d, or
d **is a factor of** n, or
d **is a divisor of** n, or
d **divides** n.

The notation **d | n** is read "d divides n." Symbolically, if n and d are integers and $d \neq 0$:

$d \mid n \Leftrightarrow \exists$an integer k such that $n = dk$.

Proof: Suppose a, b, and c are any integers such that $a \mid b$ and $a \mid c$. [*We must show that* $a \mid (b + c)$.]

By definition of divides, $b = ar$ and $c = as$ for some integers r and s. Then

$b + c = ar + as = a(r + s)$ by algebra.

Let $t = r + s$. Then t is an integer (being a sum of integers),

and thus $b + c = at$ where t is an integer. By definition of divides, then, $a \mid (b + c)$ [*as was to be shown*].

Similarly,

Suppose a, b, and c are any integers such that $a \mid b$ and $a \mid c$. [*We must show that* $a \mid (b + c)$.]

By definition of divides, $b = ar$ and $c = as$ for some integers r and s. Then

$b - c = ar - as = a(r - s)$ by algebra.

Let $t = r - s$. Then t is an integer (being a sum of integers), and thus $b - c = at$ where t is an integer. By definition of divides, then, $a \mid (b - c)$ [*as was to be shown*].

f. Use the quotient-remainder theorem with $d = 3$ to prove that the product of any three consecutive integers is divisible by 3. Use the mod notation to rewrite the result

Solution Proof: Suppose n, $n + 1$, and $n + 2$ are any three consecutive integers. *[We must show that $n(n + 1)(n + 2)$ is divisible by 3.]*

By the quotient-remainder theorem, n can be written in one of the three forms, $3q$, $3q + 1$, or $3q + 2$ for some integer q. We divide into cases accordingly.

***Case 1* (*n* = 3*q for some integer q*):** In this case,

$n(n + 1)(n + 2)$

$= 3q(3q + 1)(3q + 2)$ by substitution

$= 3 \cdot [q(3q + 1)(3q + 2)]$ by factoring out a 3.

Let $m = q(3q + 1)(3q + 2)$.

Then m is an integer because q is an integer, and sums and products of integers are integers. By substitution,

$n(n + 1)(n + 2) = 3m$ where m is an integer.

And so, by definition of divisible, $n(n + 1)(n + 2)$ is divisible by 3.

***Case 2* (*n* = 3*q* + 1 *for some integer q*):** In this case,

$n(n + 1)(n + 2)$

$= (3q + 1)((3q + 1) + 1)((3q + 1) + 2)$

by substitution

$= (3q + 1)(3q + 2)(3q + 3)$

$= (3q + 1)(3q + 2)3(q + 1)$

$= 3 \cdot [(3q + 1)(3q + 2)(q + 1)]$ by algebra.

Let $m = (3q + 1)(3q + 2)(q + 1)$. Then m is an integer because q is an integer, and sums and products of integers are integers. By substitution,

$n(n + 1)(n + 2) = 3m$ where m is an integer.

And so, by definition of divisible, $n(n + 1)(n + 2)$ is divisible by 3.

***Case 3* (*n* = 3*q* + 2 *for some integer q*):** In this case,

$n(n + 1)(n + 2)$

$= (3q + 2)((3q + 2) + 1)((3q + 2) + 2)$

by substitution

$= (3q + 2)(3q + 3)(3q + 4)$

$= (3q + 2)3(q + 1)(3q + 4)$

$= 3 \cdot [(3q + 2)(q + 1)(3q + 4)]$ by algebra

Let $m = (3q + 2)(q + 1)(3q + 4)$. Then m is an integer because q is an integer, and sums and products of integers are integers. By substitution,
$n(n + 1)(n + 2) = 3m$ where m is an integer.

And so, by definition of divisible, $n(n + 1)(n + 2)$ is divisible by 3.

In each of the three cases, $n(n + 1)(n + 2)$ was seen to be divisible by 3. But by the quotient-remainder theorem, one of these cases must occur. Therefore, the product of *any* three consecutive integers is divisible by 3.

For all integers n, $n(n + 1)(n + 2)\ mod\ 3 = 0$.

3. **Attempt *any three* of the following:** **15**

a.

i. Write the following as a single summation:

$$3\sum_{k=1}^{n}(2k-3)+\sum_{k=1}^{n}(4-5k)$$

ii. Write the following as a single product:

$$\left(\prod_{k=1}^{n}\frac{k}{k+1}\right)\cdot\left(\prod_{k=1}^{n}\frac{k+1}{k+2}\right)$$

iii. Find $1(1!) + 2(2!) + 3(3!) + \cdots + m(m!!); m = 2$

iv. Find

$$\left(\frac{1}{1+1}\right)\left(\frac{2}{2+1}\right)\left(\frac{3}{3+1}\right)\ldots\left(\frac{k}{k+1}\right); k = 3$$

v. Prove that for all nonnegative integers n and r with $r + 1 \le n$,

$$\binom{n}{r+1} = \frac{n-r}{r+1}\binom{n}{r}$$

Solution

i. $\sum_{k=1}^{n}(k-5)$

ii. $\prod_{k=1}^{n}\frac{k}{k+2}$

iii. 5

iv. ¼

v.

$$\begin{aligned}\frac{n-r}{r+1}\cdot\binom{n}{r} &= \frac{n-r}{r+1}\cdot\frac{n!}{r!(n-r)!}\\ &= \frac{n-r}{r+1}\cdot\frac{n!}{r!(n-r)\cdot(n-r-1)!}\\ &= \frac{n!}{(r+1)!\cdot(n-r-1)!}\\ &= \frac{n!}{(r+1)!\cdot(n-(r+1))!}\\ &= \binom{n}{r+1},\end{aligned}$$

b. Prove that $7^{2n} + (2^{3n-3})(3^{n-1})$ is divisible by 25 $\forall n \in N$

Solution

ution:

$2^{2n} + (7^{3n-3})(3^{n-1}) = 7^2 + (2^0) + (3^0) = 49 + 1$

$= 50$ which is divisible by 25.

$\therefore$ Result is true when n = 1

Now, let the result is true for n = k, we have

$7^{2k} + (2^{3k-3})(3^{k-1})$ is divisible by 25.

For n = k + 1 expression becomes

$7^{2(k+1)} + (2^{3(k+1)-3})(3^{(k+1)-1}) = 7^{2k+2} + 2^{3k} . 3^k = 7^{2k} . 7^2 + 2^{3k-3} . 3^{k-1}$

$= 7^{2k} . 49 + 24 . 2^{3k-3} . 3^{k-1}$

$= (50-1)7^{2k} + (25-1)2^{3k-3}\, 3^{k-1}$

$= 50 . 7^{2k} + 25 . 3^{3k-3} . 3^{k-1} - 7^{2k} - 2^{3k-3} . 3^{k-1}$

$= 50 . 7^{7^{2k}} + 25.3^{3k-3} . 3^{k-1} - (7^{2k} + 2^{3k-3}\, 3^{k-1})$

First and Second term is multiple of 25 and last term is divisible by 25 from inductive hypothesis. Therefore, expression is divisible by 25.

Hence the result is true for n = k + 1. $\therefore$ It is true for all n.

c. Determine the sequence whose recurrence relation is $a_n = 4a_{n-1} + 5a_{n-2}$ with $a_1 = 2\ and\ a_2 = 6$

Solution

$a_1 = 2$ & a_2 ...

Soln.: Here the auxiliary equation associated with recurrence rela

$a_n = 4a_{n-1} + 5a_{n-2}$ is

$x^2 = 4x + 5$

$\therefore \quad x^2 - 4x - 5 = 0$

$\therefore \quad (x+1)(x-5) = 0$

$\therefore \quad x = -1 \ \&\ x = 5$

Since the formula for the sequence is given by

$a_n = us_1^n + vs_2^n$

$\therefore a_n = u(-1)^n + v(5)^n \qquad \text{..... (1)}$

To find the value of u and v put n = 1 in (1),

$a_1 = -u + 5v \quad \therefore -u + 5v = 2 \qquad \text{..... (2)}$

Now we put n = 2 in (1),

$a_2 = u + 25v \quad \therefore u + 25v = 6 \qquad \text{..... (3)}$

$\therefore$ From equations (2) and (3) we have, $u = -\frac{2}{3}$ and $v = \frac{4}{15}$

$\therefore$ Required sequence is $a_n = \left(-\frac{2}{3}\right)(-1)^n + \left(\frac{4}{15}\right)(5)^n$

d.

i. Define G: $J_5 \times J_5 \rightarrow J_5 \times J_5$ as follows: For all $(a, b) \in J_5 \times J_5$,

$$G(a,b) = ((2a + 1)\ mod\ 5, (3b - 2)\ mod\ 5)$$

Find: $G(4,4), G(2,1),\ G(3,2), G(1,5)$

ii. Let F and G be functions from the set of all real numbers to itself. Define the product functions $F \cdot G$: **R** → **R** and $G \cdot F$: **R** → **R** as follows: For all $x \in$ **R**,

$$(F \cdot G)(x) = F(x) \cdot G(x)$$

$$(G \cdot F)(x) = G(x) \cdot F(x)$$

Does $F \cdot G = G \cdot F$? Explain.

Solution

i. (4,0), (0,1), (2,4), (3,3)

ii. $F \cdot G$ and $G \cdot F$ are equal because for all real numbers x,

$(F \cdot G)(x) = F(x) \cdot G(x)$ by definition of $F \cdot G$

$= G(x) \cdot F(x)$ by the commutative law for multiplication of real numbers

$= (G \cdot F)(x)$ by definition of $G \cdot F$.

e.

i. Define Floor: $\mathbf{R} \to \mathbf{Z}$ by the formula $Floor(x) = \lfloor x \rfloor$, for all real numbers x.

- Is Floor one-to-one? Prove or give a counterexample.
- Is Floor onto? Prove or give a counterexample.

ii. Let S be the set of all strings of 0's and 1's, and define

$l: S \to \mathbf{Z}^{nonneg}$ by

$l(s) =$ the length of s, for all strings s in S.

- Is l one-to-one? Prove or give a counterexample.
- Is l onto? Prove or give a counterexample.

Solution

i. $\lfloor x \rfloor =$ that unique integer n such that $n \leq x < n + 1$.

- *Floor is not one-to-one:*

  Floor(0) $= 0 =$ Floor $(1/2)$ but $0 \neq 1/2$.

- *Floor is onto:* Suppose $m \in \mathbf{Z}$. *[We must show that there exists a real number y such that Floor (y)=m.]* Let $y = m$.

  Then Floor $(y) =$ Floor $(m) = m$ since m is an integer.
  (Actually, Floor takes the value m for *all* real numbers in the interval $m \leq x < m + 1$.) Hence there exists a real number y such that Floor $(y) = m$.

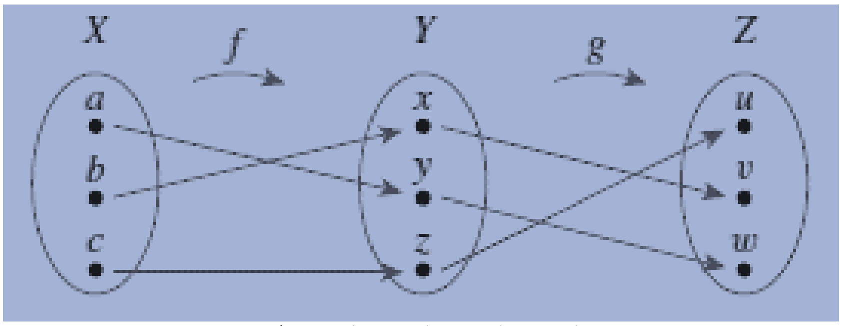
ii.

l is not one-to-one: $l(0) = l(1) = 1$ but $1 \neq 0$.
l is onto: Suppose n is a nonnegative integer. *[We must show that there exists a string s in S such that $l(s) = n$.]* Let

$$s = \begin{cases} \epsilon \text{ (the null string)} & \text{if } n = 0 \\ \underbrace{00\ldots0}_{n\ 0\text{'s}} & \text{if } n > 0 \end{cases}.$$

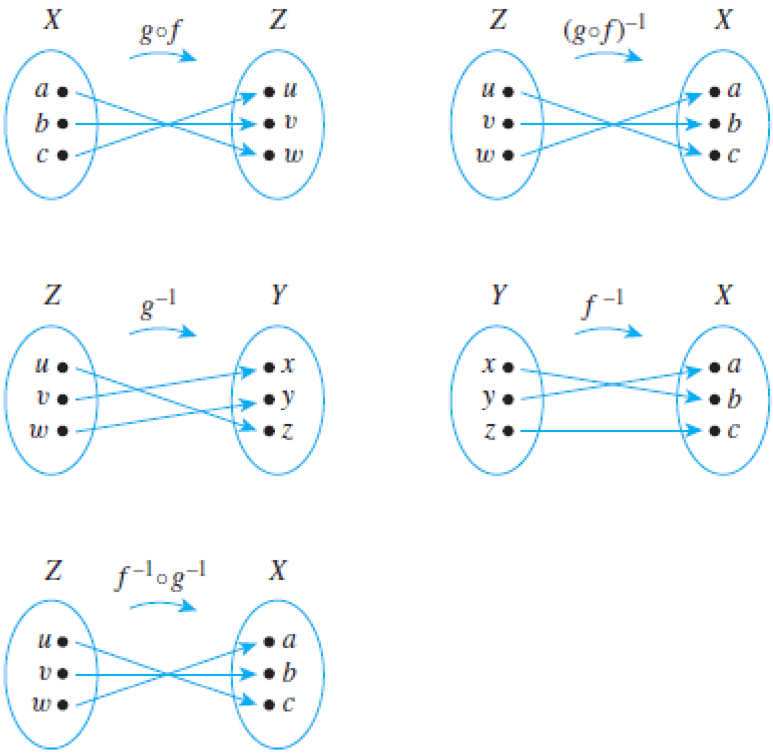
Then $l(s) =$ the length of $s = n$. This is what was to be shown.

f. Let $X = \{a, c, b\}$, $Y = \{x, y, z\}$, and $Z = \{u, v, w\}$. Define $f: X \to Y$ and $g: Y \to Z$ by the arrow diagrams below.



Find: $g \ o \ f, (g \ o \ f)^{-1}, \ f^{-1}, \ g^{-1}, \ f^{-1} o \ g^{-1}$.
$How \ (g \ o \ f)^{-1} \ and \ f^{-1} o \ g^{-1} \ are \ related?$

Solution



The functions $(g \circ f)^{-1}$ and $f^{-1} \circ g^{-1}$ are equal.

4. **Attempt *<u>any three</u>* of the following:** **15**

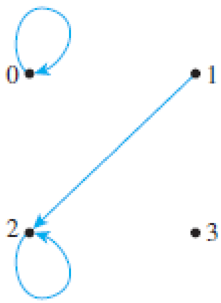
a Draw the directed graph for the following relations:

i. A relation $R \ on \ A = \{0, 1, 2, 3\}$ by $R = \{(0, 0), (1, 2), (2, 2)\}$.

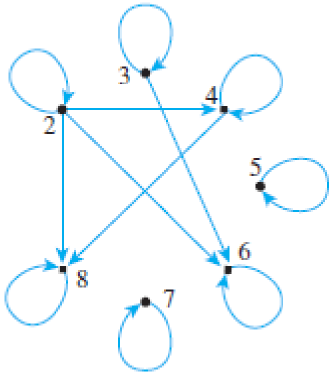
ii. Let A = {2, 3, 4, 5, 6, 7, 8} and define a relation $R \ on \ A$ as follows:

$For \ all \ x, y \in A, x \ R \ y \Leftrightarrow x \mid y.$

Solution i.



ii.



b Determine whether the following relations are reflexive, symmetric, transitive or none of these. Justify your answer.

i. R is the "greater than or equal to" relation on the set of real numbers: $For\ all\ x, y \in \mathbf{R}, x\ R\ y \Leftrightarrow x \geq y.$

ii. D is the relation defined on R as follows: $For\ all\ x, y \in R, x\ D\ y \Leftrightarrow xy \geq 0.$

Solution

i. ***R is reflexive:*** R is reflexive ⇔ for all real numbers x,x R x. By definition of R, this means that for all real numbers x, $x \geq x$. In other words, for all real numbers x, $x > x$ or $x = x$. But this is true.

R is not symmetric: R is symmetric ⇔ for all real numbers x and y, if x R y then y R x. By definition of R, this means that for all real numbers x and y, if $x \geq y$ then $y \geq x$. But this is false. As a counterexample, take $x = 1$ and $y = 0$. Then $x \geq y$ but y *not* $\geq x$ because $1 \geq 0$ but 0 *not* ≥ 1.

R is transitive: R is transitive ⇔ for all real numbers x, y, and z, if x R y and y R z then x R z. By definition of R, this means that for all real numbers x, y and z, if $x \geq y$ and $y \geq z$ then $x \geq z$. But this is true by definition of $\geq$ and the transitive property of order for the real numbers.

ii. ***D is reflexive:*** For D to be reflexive means that for all real numbers x,x D x. But by definition of D, this means that for all real numbers x, $xx = x2 \geq 0$, which is true.

D is symmetric: For D to be symmetric means that for all real numbers x and y, if x D y then y D x. But by definition of D, this means that for all real numbers x and y, if $xy \geq 0$ then $yx \geq 0$, which is true by the commutative law of multiplication.

D is not transitive: For D to be transitive means that for all real numbers x, y, and z, if x D y and y D z then x D z. By definition of D, this means that for all real numbers x, y, and z, if $xy \geq 0$ and $yz \geq 0$ then $xz \geq 0$. But this is false: there exist real numbers x, y, and z such that $xy \geq 0$ and $yz \geq 0$ but xz *not* ≥ 0. As a counterexample, let $x = 1$, $y = 0$, and $z = -1$. Then x D y and y D z because $1 \cdot 0 \geq 0$ and $0 \cdot (-1) \geq 0$. But x _D z because $1 \cdot (-1)$ *not* ≥ 0.

c Let **R** be the set of all real numbers and define a relation R on $\mathbf{R} \times \mathbf{R}$ as follows: For all (a, b) and (c, d) in $\mathbf{R} \times \mathbf{R}$, $(a, b)\ R\ (c, d) \Leftrightarrow$ either $a < c$ or both $a = c$ and $b \leq d$.

Is R a partial order relation? Prove or give a counterexample.

Solution R is a partial order relation.

Proof:

***R* is reflexive:** Suppose $(a, b) \in \mathbf{R} \times \mathbf{R}$. Then $(a, b)\ R\ (a, b)$ because $a = a$ and $b \leq b$.

***R* is antisymmetric:** Suppose (a, b) and (c, d) are ordered pairs of real numbers such that $(a, b)\ R\ (c, d)$ and $(c, d)\ R\ (a, b)$. Then

either $a < c$ or both $a = c$ and $b \leq d$

and

either $c < a$ or both $c = a$ and $d \leq b$.

Thus

$$a \leq c \text{ and } c \leq a$$

and so

$$a = c.$$

Consequently,

$$b \leq d \quad \text{and} \quad d \leq b$$

and so

$$b = d.$$

Hence $(a, b) = (c, d)$.

***R* is transitive:** Suppose (a, b), (c, d), and (e, f) are ordered pairs of real numbers such that $(a, b)\ R\ (c, d)$ and $(c, d)\ R\ (e, f)$. Then

either $a < c$ or both $a = c$ and $b \leq d$

and

either $c < e$ or both $c = e$ and $d \leq f$.

It follows that one of the following cases must occur.

Case 1 ($a < c$ and $c < e$): Then by transitivity of $<$, $a < e$, and so $(a, b)\ R\ (e, f)$ by definition of R.

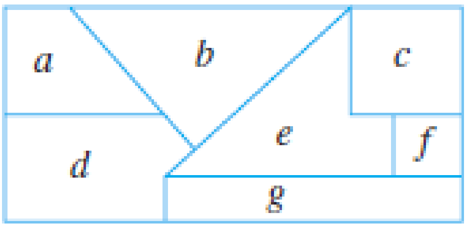
Case 2 ($a < c$ and $c = e$): Then by substitution, $a < e$, and so $(a, b)\ R\ (e, f)$ by definition of R.

Case 3 ($a = c$ and $c < e$): Then by substitution, $a < e$, and so $(a, b)\ R\ (e, f)$ by definition of R.

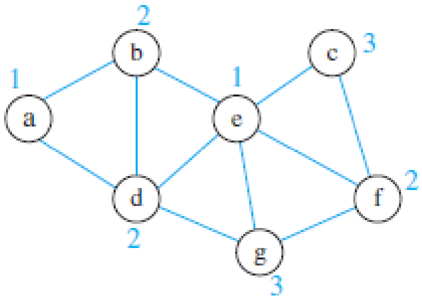
Case 4 ($a = c$ and $c = e$): Then by definition of R, $b \leq d$ and $d \leq f$, and so by transitivity of $\leq$, $b \leq f$. Hence $a = e$ and $b \leq f$, and so $(a, b)\ R\ (e, f)$ by definition of R.

In each case, $(a, b)\ R\ (e, f)$. Therefore, R is transitive. Since R is reflexive, antisymmetric, and transitive, R is a partial order relation.

d Imagine that the diagram shown below is a map with countries labeled a–g. Is it possible to color the map with only three colors so that no two adjacent countries have the same color? To answer this question, draw and analyze a graph in which each country is represented by a vertex and two vertices are connected by an edge if, and only if, the countries share a common border.

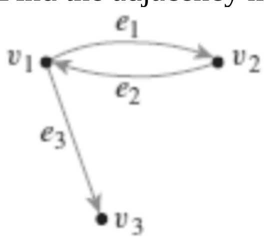


Solution



Vertex e has maximal degree, so color it with color #1. Vertex a does not share an edge with e, and so color #1 may also be used for it. From the remaining uncolored vertices, all of d, g, and f have maximal degree. Choose any one of them, say d, and use color #2 for it. Observe that vertices b, c, and f do not share an edge with d, but c and f share an edge with each other, which means that color #2 may be used for only one of c or f . So color b with color #2, and choose to color f with color #2 because the degree of f is greater than the degree of c. From the remaining uncolored vertices, g has maximal degree, so color it with color #3. Then observe that because g does not share an edge with c, color #3 may also be used for c. At this point, all vertices have been colored.

e i. Find the adjacency matrix of the following graph:



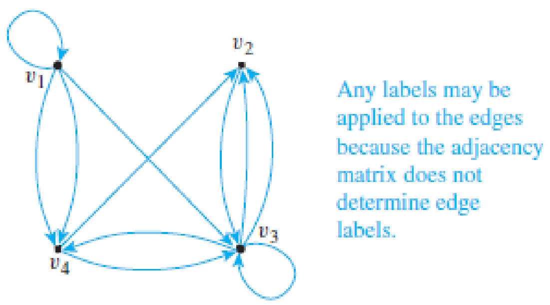
ii. Find directed graphs that have the following adjacency matrix:

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Solution i.

	v_1	v_2	v_3
v_1	0	1	1
v_2	1	0	0
v_3	0	0	0

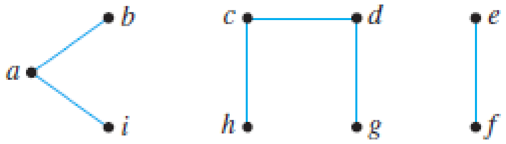
ii.



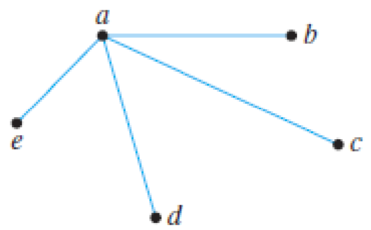
f For the following either draw the graph as per the specifications or explain why no such graph exists:

i. Graph, circuit-free, nine vertices, six edges
ii. Tree, six vertices, total degree 14
iii. Tree, five vertices, total degree 8
iv. Graph, connected, six vertices, five edges, has a nontrivial circuit
v. Graph, two vertices, one edge, not a tree

Solution
i. One such graph is



ii. There is no tree with six vertices and a total degree of 14. Any tree with six vertices has five edges and hence a total degree of 10, not 14.
iii. One such tree is shown.



iv. No such graph exists. A connected graph with six vertices and five edges is a tree. Hence such a graph cannot have a nontrivial circuit.

v.


5. Attempt *__any three__* of the following: **15**

a. There are four bus lines between A and B and three bus lines between B and C. In how many ways can a man travel
i. by bus from A to C by way of B?
ii. round-trip by bus from A to C by way of B?
iii. round-trip by bus from A to C by way of B if he does not want to use a bus line more than once?

Solution
i. 12
ii. 144
iii. 72

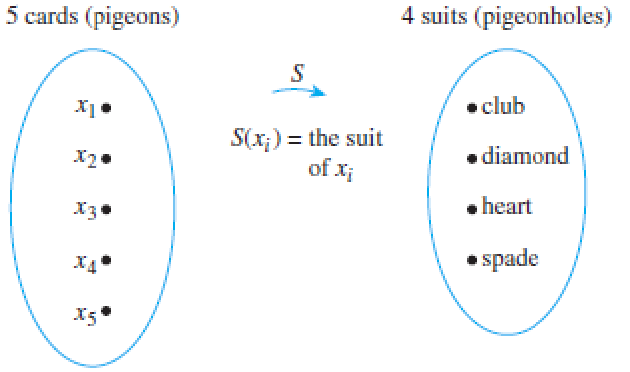
b.
i. How many ways can the letters of the word ALGORITHM be arranged in a row? **9! = 362, 880.**
ii. How many ways can the letters of the word ALGORITHM be arranged in a row if A and L must remain together (in order) as a unit? **8! = 40320**
iii. How many ways can three of the letters of the word ALGORITHM be selected and written in a row? **504**
iv. How many ways can six of the letters of the word ALGORITHM be selected and written in a row if the first letter must be A? **6720**
v. How many ways can the letters of the word ALGORITHM be arranged in a row if the letters GOR must remain together (in order) as a unit? **7! = 5040**

c.
i. If 4 cards are selected from a standard 52-card deck, must at least 2 be of the same suit? Why?

No. For instance, the aces of the four different suits could be selected.

ii. If 5 cards are selected from a standard 52-card deck, must at least 2 be of the same suit? Why?

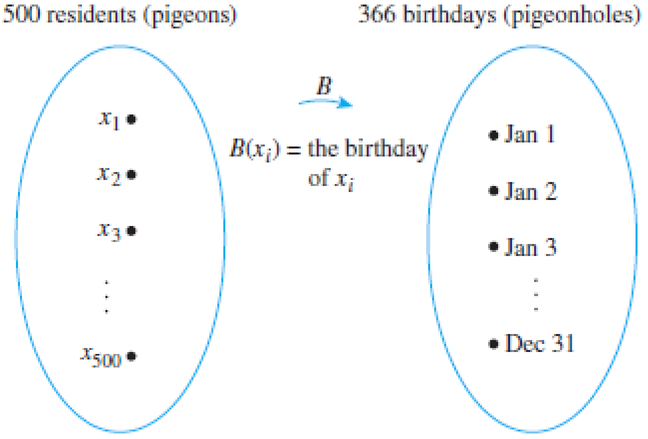
Yes. Let x_1, x_2, x_3, x_4, x_5 be the five cards. Consider the function S that sends each card to its suit.



By the pigeonhole principle, S is not one-to-one: $S(x_i) = S(x_j)$ for some two cards x_i and x_j. Hence at least two cards have the same suit.

iii. A small town has only 500 residents. Must there be 2 residents who have the same birthday? Why?

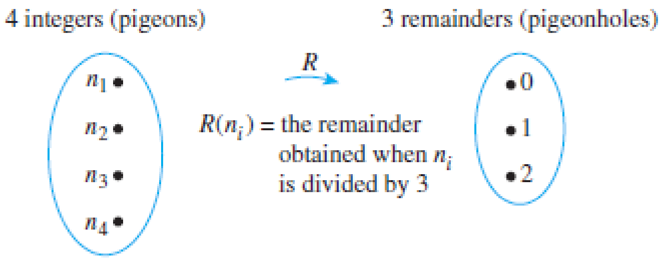
Yes. Denote the residents by $x_1, x_2, \ldots, x_{500}$. Consider the function B from residents to birthdays that sends each resident to his or her birthday:



By the pigeonhole principle, B is not one-to-one: $B(x_i) = B(x_j)$ for some two residents x_i and x_j. Hence at least two residents have the same birthday.

iv. Given any set of four integers, must there be two that have the same remainder when divided by 3? Why?

Yes. There are only three possible remainders that can be obtained when an integer is divided by 3: 0, 1, and 2. Thus, by the pigeonhole principle, if four integers are each divided by 3, then at least two of them must have the same remainder. More formally, call the integers $n1$, $n2$, $n3$, and $n4$, and consider the function R that sends each integer to the remainder obtained when that integer is divided by 3:



By the pigeonhole principle, R is not one-to-one, $R(n_i) = R(n_j)$ for some two integers n_i and n_j. Hence at least two integers must have the same remainder.

v. Given any set of three integers, must there be two that have the same remainder when divided by 3? Why?

No. For instance, {0, 1, 2} is a set of three integers no two of which have the same remainder when divided by 3.

d.

i. How many distinguishable ways can the letters of the word *HULLABALOO* be arranged in order?

$$\frac{10!}{2!1!1!3!2!1!} = 151{,}200$$

ii. How many distinguishable orderings of the letters of HULLABALOO begin with U and end with L?

$$\frac{8!}{2!1!1!2!2!} = 5{,}040$$

iii. How many distinguishable orderings of the letters of HULLABALOO contain the two letters HU next to each other in order?

$$\frac{9!}{1!2!1!3!2!} = 15{,}120$$

e. A bakery produces six different kinds of pastry, one of which is eclairs. Assume there are at least 20 pastries of each kind.

i. How many different selections of twenty pastries are there? **53130**

ii. How many different selections of twenty pastries are there if at least three must be eclairs? **26,334**

iii. How many different selections of twenty pastries contain at most two eclairs? **26, 796**

f. A drug-screening test is used in a large population of people of whom 4% actually use drugs. Suppose that the false positive rate is 3% and the false negative rate is 2%. Thus a person who uses drugs tests positive for them 98% of the time, and a person who does not use drugs tests negative for them 987% of the time.

i. What is the probability that a randomly chosen person who tests positive for drugs actually uses drugs?

ii. What is the probability that a randomly chosen person who tests negative for drugs does not use drugs?

Let A be the event that a randomly chosen person tests positive for drugs, let B_1 be the event that a randomly chosen person uses drugs, and let B_2 be the event that a randomly chosen person does not use drugs. Then A^c is the event that a randomly chosen person does not test positive for drugs, and $P(B_1) = 0.04$, $P(B_2) = 0.96$, $P(A \mid B_2) = 0.03$, and $P(A^c \mid B_1) = 0.02$. Hence $P(A \mid B_1) = 0.97$ and $P(A^c \mid B_2) = 0.98$.

a. $$P(B_1 \mid A) = \frac{P(A \mid B_1)P(B_1)}{P(A \mid B_1)P(B_1) + P(A \mid B_2)P(B_2)} = \frac{(0.97)(0.04)}{(0.97)(0.04) + (0.03)(0.96)} \cong 57.4\%$$

b. $$P(B_2 \mid A^c) = \frac{P(A^c \mid B_2)P(B_2)}{P(A^c \mid B_1)P(B_1) + P(A^c \mid B_2)P(B_2)} = \frac{(0.98)(0.96)}{(0.02)(0.04) + (0.98)(0.96)} \cong 99.9\%$$
